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Some asymptotic results in the theory of wave-energy devices

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Summary

Evans [2] has shown how the solution of the radiation problem for water-waves gives all the necessary parameters for determining the efficiency of a wave-energy device. We consider how this radiation problem can be solved in certain limits for several types of device. First, a moving-cylinder device which is many wavelengths below the surface is examined, using Leppington and Siew's [11] method. We then calculate the efficiency of a Cockerell raft device which is either many wavelengths, or only a small fraction of a wavelength, long. Finally, the problem of a thin, symmetric gas-filled bag in a wave field is solved and the first-order solution given for the efficiency.

1. Introduction

There has been considerable interest recently in the theory of wave-energy devices. Linearised water-wave theory seems to give very good agreements with experiments, and so calculations based on classical ship dynamics can be made.

Evans [2] and Mei [13] have both independently produced a theory for the absorption of power from an incident wave train by a moving body. In particular, Evans [2] derived the efficiency of power absorption when the body is two-dimensional and is constrained to move in one or a combination of two modes. He finds that the efficiency depends only on parameters which can be calculated from the radiation problem, where the body generates waves by moving in these given modes.

Using this theory, precise expressions for the efficiency of a rolling vertical plate can be derived and are given in a note by a reviewer in [7]. This is because the solution of the associated radiation problem is known exactly [15]. Unfortunately, this is one of the very few exact solutions available, and numerical methods are necessary for the heaving or swaying half-immersed circular cylinder which were also considered in [7]. The relevant constants have been calculated by Frank [3] and these were used to compute the efficiency of a three-dimensional floating hemisphere which is heaving, using computations by Havelock [5]. A particularly interesting result is that if the two-dimensional cylinder is made to oscillate in the correct combination of two modes, then in theory 100% efficiency can be obtained.

Another type of two-dimensional wave-energy absorber which has been proposed is the Cockerell raft. Originally, it was thought of as a large number of rafts all hinged together in line with the incoming waves. As the rafts moved relative to each other, power could be

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taken off as the hinges moved. Experiments with this type of configuration proved disappointing, though, and so new experiments are being made on pairs of hinged plates connected in parallel down a long spine. Computations suggest that this type of raft may have an efficiency comparable with that of the Salter duck, which has one of the highest efficiencies known for a practical device. The lengths and masses of the two sections of each raft are critical, however, so detailed calculations must be made to find the best design for a given wavelength of the sea.

The above two devices rely, as do many others, on the motion of the bodies in the incident wave field for the transfer of energy. Some devices, however, consist of flexible bags attached to fixed structures. As the surface of the water rises and falls, the air in the bags expands and contracts, driving a turbine to generate power. This kind of mechanism is used mainly in so-called attenuating devices, where the energy is gradually absorbed along a long body, rather than stopped at a point. One such device is the French flexible bag device, where a whole series of air-filled containers is moored to a raft which faces into the waves. A description of this converter and a simple theoretical model of it are given by French [4].

There are many other types of wave-energy devices and each of them presents different theoretical difficulties. A review of most of the current designs being considered in the U.K., including those described above, are given in [1].

In this paper, some results are obtained for the efficiency of these converters in certain limits of the relevant parameters. First, the results of Leppington and Siew [11] can be used for fully submerged circular cylinders in the limit $k(h-a)^2/a \to \infty$, where k is the wave-number of the incident wave, a the radius of the cylinder and h its depth. In this limit they show that the surface effect can be ignored and to leading order the potential is equal to its far-field value with a correction for the boundary condition on the circle. Thus a leading-order analysis of the radiation problem can be carried out, and the efficiency found using Evans' [2] results.

Similarly, in the limit $kl \to \infty$ where l is the length of the device, the Cockerell raft may be modelled using a matched asymptotic expansion scheme due to Leppington [10] for finite docks. In principle, all the necessary parameters can be calculated for the radiation problem as a series in $\epsilon = (kl)^{-1}$. It is also possible to calculate the low-frequency limit $kl \to 0$ by using an integral-equation formulation and then solving this by the method of successive approximations as a series in $\delta = kl$. A check on this latter result can be obtained from a matching scheme, which confirms the first few terms in the series.

Finally, using a method developed elsewhere [14], the case of a plate making small oscillations with a given shape about the vertical is considered. The theory can be extended to symmetric bodies which pulsate with a given surface shape, and which can be used as a model for a single infinitely long flexible bag, aligned perpendicularly to the incoming waves. Note that this is not the same as the French flexible bag device, where the bags are in line with the incoming waves. The results are useful, however, because they contain the shape of the bags. All the other analyses of the French device ignore this factor, so the present theory may be useful in determining the ideal bag shape.

2. Formulation

We restrict ourselves to two-dimensional irrotational flows of inviscid incompressible fluid. Cartesian axes are chosen with the y-axis vertically upwards and with $y = 0$ at the undisturbed free surface, while x is horizontally to the right. The usual assumptions of linearised water-wave theory are assumed to hold, so that we can define a velocity potential ϕ satisfying

$$\nabla^2\phi = 0 \quad (y<0), \tag{2.1}$$

and the linearised free-surface condition

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0 \quad (y=0). \tag{2.2}$$

A simple harmonic wave is normally incident on the body from $x = -\infty$, with a wavenumber k and frequency $\omega = (kg)^{1/2}$. Henceforth we shall suppress the time-dependence of the potential,

$$\phi(x, y, t) = \operatorname{Re}\Phi(x, y)\,e^{-i\omega t},$$

so that the incident wave is given by

$$\Phi_{\text{inc}} = \exp(ikx + ky).$$

The body is taken to be an infinitely long cylinder with horizontal generators parallel to the wave crests. It is constrained to make small motions in one of three modes: heave, sway or roll. It has been shown by Evans [2] and Mei [13] that all the information required to find the efficiency of this device is contained in the simpler radiation problem, according to the formula explained below. Suppose Φ_j is the solution of the following radiation problem for swaying, heaving or rolling motions ($j = 1$, 2 or 3),

$$\begin{aligned} &\frac{\partial\Phi_j}{\partial n} = n_j \quad \text{on the cylinder}, \qquad \frac{\partial\Phi_j}{\partial n} + k\Phi_j = 0 \quad (y=0), \\ &\Phi_j \sim A_j^+\, e^{-ikx+ky} \quad (x \to +\infty), \\ &\Phi_j \sim A_j^-\, e^{ikx+ky} \quad (x \to -\infty), \end{aligned} \tag{2.3}$$

and let

$$\delta = |A_j^-|^2 / \left(|A_j^+|^2 + |A_j^-|^2\right). \tag{2.4}$$

The constants $A_j^{\pm}$ are eigenvalues of the problem, to be found. Also, (n_1, n_2) is the normal to the body and $n_3 = n_2 x - (y+c)n_1$, where c is the depth of the point of rotation. If power is taken off the body by a spring and damper system obeying the force law

$$F = D\dot{\zeta} + K\zeta, \tag{2.5}$$

where D and K are the damper and spring constant, respectively, then the efficiency is given, according to Evans [2], by

$$E = \frac{4\omega^2 D b_{jj}(1-\delta)}{\left\{K - (m + a_{jj})\omega^2\right\}^2 + \omega^2(b_{jj} + D)^2}. \tag{2.6}$$

The constants a_{jj} and b_{jj} are the added mass and radiation damping of the problem (2.3), while $k = \omega^2/g$.

For a given cylinder, the maximum efficiency E_M can be obtained only if the system is correctly tuned:

$$E = E_M = 1 - \delta$$

if

$$K = \{m + a_{jj}(\omega_0)\}\omega_0^2, \qquad D = b_{jj}(\omega_0). \tag{2.7}$$

One more factor that has to be taken into account is the bandwidth. Although the system may be tuned for incident waves at a frequency ω_0, slight changes in ω may drastically reduce the efficiency. Some measure of how much frequency variation is possible is therefore necessary in practice.

3. Totally immersed circular cylinder

We can make use of the results of Leppington and Siew [11] to calculate the constants $A_j^{\pm}$, a_{jj} and b_{jj} as functions of k for a circular cylinder of radius a and depth h. In the scattering problem, Leppington and Siew [11] found that, to leading order in the limit $k(h-a)^2/a \gg 1$, good results were obtained by taking the potential to be equal to its far-field value, corrected only for the boundary condition on the circle. No correction at first order was necessary for the free-surface effects. We can follow a similar procedure in the radiation problem, but its validity remains to be proved.

Consider the circular cylinder in Fig. 1 and choose Cartesian axes with the origin at the centre of the circle and the surface at $y = h$. Also, define the complex variable $z = x + \mathrm{i}y$.

Scale the dimension a out of the problem by the transformation $(a, h/a, z/a, ka) \to (1, h, z, k)$. At a point $z = \mathrm{e}^{\mathrm{i}\theta}$ on the circle $\mathscr{C}$, $\phi_n = \sin\theta$ for heaving motions, while for swaying motions $\phi_n = \cos\theta$.

Consider first heaving motions, in which case the problem is even in x and we need consider only half the (x, y) plane. Subtract the far-field potential from the total,

$$\Phi = \phi\,\mathrm{e}^{-kh} + A\,\mathrm{e}^{k(\mathrm{i}x+y)-kh}, \tag{3.1}$$

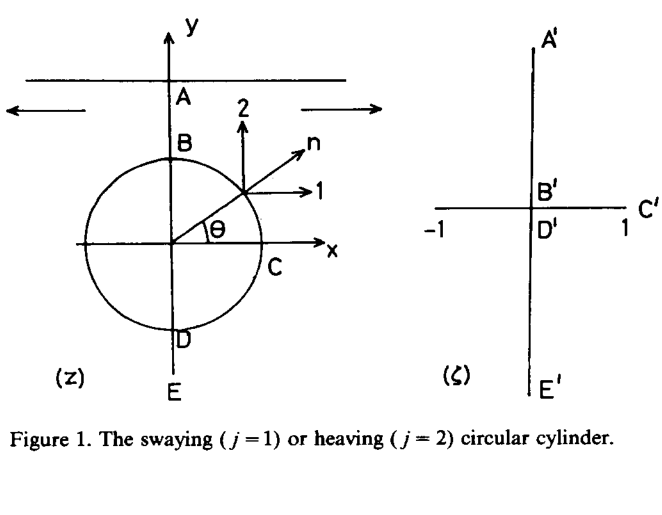


Figure 1. The swaying ($j = 1$) or heaving ($j = 2$) circular cylinder.

where here $e^{-kh}A = A_2^+ = A_2^-$ by symmetry. Then the problem for ϕ is

$$\nabla^2\phi = 0 \quad (y < h), \tag{3.2a}$$

$$\phi_y - k\phi = 0 \quad (y = h), \tag{3.2b}$$

$$\frac{\partial\phi}{\partial n} = \sin\theta - ikA\, e^{-i\theta}\exp(ik\, e^{-i\theta}) \quad (\text{on } \mathscr{C}), \tag{3.2c}$$

$$\frac{\partial\phi}{\partial x} = -ikA\, e^{ky} \quad (x = 0). \tag{3.2d}$$

Following [11], we take as a first approximation a potential that satisfies all of these conditions except the free-surface condition (3.2b). This is replaced by the simpler condition $\phi_y = 0$ on $y = h$. To find this, use the conformal mapping

$$2\zeta = z + \frac{1}{z}. \tag{3.3}$$

This maps the region to the right of $x = 0$ and of the right half of the circle onto the right-hand ζ-plane outside the interval $\zeta = (\xi, 0)$, $0 > \xi > 1$. We can solve this problem by using a simple Green function with zero normal derivative on the lines $\xi = 0$ and $\eta = 0$,

$$G = \frac{1}{2\pi}\log|(\zeta - \zeta_1)(\zeta - \bar{\zeta}_1)(\zeta + \zeta_1)(\zeta + \bar{\zeta}_1)|. \tag{3.4}$$

The solution is then

$$\phi(\zeta_1) = -ikA\int_{-\infty}^{H}\exp\{kt(\eta)\}G(\zeta_1, 0, \eta)d\eta$$
$$+2\int_0^1\left[(1-\xi^2)^{1/2} - Af(\xi)\right]G(\zeta_1, \xi, 0)d\xi,$$

$$f(\xi) = ik\xi\exp(ik\xi)\sinh\left\{k(1-\xi^2)^{1/2}\right\}$$
$$+k(1-\xi^2)^{1/2}\exp(ik\xi)\cosh\left\{k(1-\xi^2)^{1/2}\right\}, \tag{3.5}$$

$$H = (h^2 - 1)/(2h), \qquad \eta = (t^2 - 1)/(2t).$$

Now the expression (3.5) is required to tend to zero as $\zeta_1 \to \infty$, while as it stands it grows logarithmically. This gives a condition from which the eigenvalue A can be found, and, collecting logarithmically large terms as $\zeta_1 \to \infty$,

$$A\left[ik\int_{-\infty}^{H}\exp\{kt(\eta)\}d\eta + 2\int_0^1 f(\xi)d\xi\right] = 2\int_0^1(1-\xi^2)^{1/2}d\xi. \tag{3.6}$$

We also require the added-mass and damping coefficients, which can be found by calculating the total resisting force on the cylinder. First, there is the force due to the incident-wave part of Φ,

$$F_1 = \rho\int_{-\pi/2}^{\pi/2}\sin\theta\exp(ik\, e^{-i\theta})d\theta. \tag{3.7}$$

Then we must add the contribution from the correction in (3.5), which is given by

$$F_2 = \rho\int_{-\pi/2}^{\pi/2} \sin\theta\, \phi(\cos\theta, \sin\theta)\mathrm{d}\theta$$

$$= \frac{\rho}{\pi}\int_0^1\left\{-\mathrm{i}k\int_{-\infty}^{H} \exp\{kt(\eta)\}\log(\xi^2+\eta^2)\mathrm{d}\eta\right.$$

$$\left. +2\int_0^1\left[(1-\xi_1^2)^{1/2} - Af(\xi_1)\right]\log|\xi^2-\xi_1^2|\mathrm{d}\xi_1\right\}\mathrm{d}\xi. \tag{3.8}$$

If the total force is $\mathrm{e}^{-kh}F = F_1 + F_2$, then the added-mass term is 180° out of phase with the acceleration, and is thus the real part of F. Similarly, the damping is 180° out of phase with the velocity and is the imaginary part,

$$a_{22} = \mathrm{Re}\, F, \qquad b_{22} = \mathrm{Im}\, F. \tag{3.9}$$

Suppose now the cylinder is swaying. The calculations are all very similar, except that now ϕ_n is $\cos\theta$ rather than $\sin\theta$ as above. A more suitable Green function for this problem is

$$G_2 = \frac{1}{2\pi}\log\left|\frac{(\zeta-\zeta_1)(\zeta-\bar{\zeta}_1)}{(\zeta+\zeta_1)(\zeta+\bar{\zeta}_1)}\right|, \tag{3.10}$$

rather than (3.4), and using this the solution is

$$\phi(\zeta_1) = -A\int_{-\infty}^{H}\exp\{kt(\eta)\}G_{2\eta}(\zeta_1, 0, \eta)\mathrm{d}\eta$$

$$+\int_0^1[\xi - 2Af(\xi)]G_2(\zeta_1, \xi, 0)\mathrm{d}\xi. \tag{3.11}$$

The constant A is found by requiring the $O(1)$ terms to vanish as $\zeta_1 \to \infty$, so that

$$A\left[\int_{-\infty}^{H}\exp\{kt(\eta)\}\mathrm{d}\eta + 2\int_0^1 f(\xi)\mathrm{d}\xi\right] = \int_0^1\xi\mathrm{d}\xi = \tfrac{1}{2}. \tag{3.12}$$

Then, by symmetry, $\mathrm{e}^{-kh}A = A_1^+ = -A_1^-$. Finally, the total force is $\mathrm{e}^{-kh}F = F_1 + F_2$, where

$$F_1 = \rho\int_{-\pi/2}^{\pi/2}\cos\theta\exp(\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}\theta})\mathrm{d}\theta, \tag{3.13a}$$

$$F_2 = \rho\int_{-\pi/2}^{\pi/2}\cos\theta\, \phi(\cos\theta, \sin\theta)\mathrm{d}\theta$$

$$= \frac{\rho}{\pi}\int_0^1\xi\left\{-2\mathrm{i}A\int_{-\infty}^{H}\exp\{kt(\eta)\}\frac{\eta\mathrm{d}\eta}{\xi^2+\eta^2}\right.$$

$$\left. +\int_0^1[\xi_1 - 2Af(\xi_1)]\log\left|\frac{\xi-\xi_1}{\xi+\xi_1}\right|\mathrm{d}\xi_1\right\}\frac{\mathrm{d}\xi}{(1-\xi^2)^{1/2}}. \tag{3.13b}$$

4. The Cockerell raft

In this section we consider, as a model for the Cockerell wave-energy device, a floating raft in two sections which move independently. The relevant radiation problem, in which the motion of the plates generates waves, can be solved explicitly in two limits, which we now consider.

(a) The short-wave limit

Asymptotic solutions in this limit were first given by Holford [6], who showed how the problem could be formulated as an integral equation of the second kind. When this is done correctly, the kernel is sufficiently small to allow solution by iteration. This gives a mathematically rigorous leading-order term, but is very cumbersome when it comes to calculating higher-order terms. Leppington [8,9] extended the method to give these higher-order corrections, and Leppington [10] gives another method based on matched asymptotic expansions. A model problem for a two-dimensional dock is solved where the velocity on the dock is uniform along its length, and it is this method we use here for a more general hinged raft.

In order to illustrate the method and to simplify the details, consider first a symmetric raft as in Fig. 2. The ends are resting on the surface while the hinge at the centre allows the plates to move, giving a velocity distribution

$$\frac{\partial \phi}{\partial y} = V(x) = \begin{cases} x & (1 > x > 0) \\ -x & (0 > x > -1) \end{cases} \quad (y = 0). \tag{4.1}$$

Here we have scaled the length l of the barrier out of the problem, which has the effect of replacing kl by k and l by 1 everywhere. Let $\epsilon = 1/(kl)$, so that in this limit $\epsilon \to 0$.

In problems with sharp edges, the potential ϕ must satisfy an edge condition, which ensures uniqueness and guarantees conservation of energy. This states that near the ends of the raft at $z = \pm 1$, $\phi \sim (z \pm 1)^{-\gamma}$ where $\gamma < 1$. A first approximation, which can be expected to be good many wavelengths away from the free surface, is obtained by simply putting $\epsilon = 0$ in the problem, which gives

$$\nabla^2 \phi_0 = 0,$$
$$\frac{\partial \phi_0}{\partial y} = V(x) \quad (|x| < 1), \qquad \phi_0 = 0 \quad (|x| > 1,\ y = 0). \tag{4.2}$$

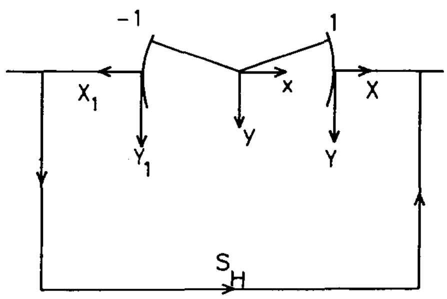


Figure 2. The symmetric dock.

The edge condition will not necessarily be satisfied by ϕ_0 since the edges are close to the free surface, where this solution is not valid (although in this particular case it is, as we shall see below). The problem (4.2) has no wavy solutions, so at infinity $\phi_0 \to 0$.

It is usually more convenient to work with the complex potential, which is

$$w_0(\zeta_1) = \frac{1}{2\pi}\int_0^\infty \tilde{V}(\eta)\log\frac{\eta + i\zeta_1}{\eta - i\zeta_1}\,d\eta,$$

where

$$\tilde{V}(\eta) = \begin{cases} 8\eta(1+\eta^2)^{-3} & (\eta > 1), \\ 8\eta^3(1+\eta)^{-3} & (0 < \eta < 1). \end{cases} \tag{4.3}$$

The approximation ϕ_0 is not good at points near the edges at $x = \pm 1$. However, an observer at either edge will be relatively unaware of the other edge, as $\epsilon \to 0$. Consider therefore a small region near $x = 1$ defined by the coordinates

$$x = 1 + \epsilon X, \qquad y = \epsilon Y, \tag{4.4}$$

and let

$$\phi(x, y) = \phi(1 + \epsilon X, \epsilon Y) = \Phi(X, Y). \tag{4.5}$$

Similarly, near $x = -1$, define $x = -1 - \epsilon X_1$, $y = \epsilon Y_1$ and $\phi(x, y) = \Pi(X_1, Y_1)$. In this case, by the symmetry of the problem, $\Phi = \Pi$ so we need only consider the right inner region in what follows. The boundary-value problem for Φ is now the semi-infinite dock problem considered by Holford [6]:

$$\begin{aligned} &\nabla^2\Phi = 0, \\ &\frac{\partial \Phi}{\partial Y} = \epsilon V(1 + \epsilon X) \quad (Y = 0,\ -2/\epsilon < X < 0), \\ &\Phi + \frac{\partial \Phi}{\partial Y} = 0 \quad (Y = 0,\ X > 0 \text{ or } X < -2/\epsilon). \end{aligned} \tag{4.6}$$

If we now define

$$R = (X^2 + Y^2)^{1/2}, \qquad \delta = \epsilon R, \tag{4.7}$$

then the matching condition that ensures a smooth transition between the regions is

$$\lim_{\delta \to 0}\phi(\delta, \theta) = \lim_{R \to \infty}\Phi(R, \theta). \tag{4.8}$$

Holford [6] has shown that the inner solution has the following form

$$\begin{aligned} &\Phi(R, \theta) \sim \epsilon^{1/2} I R^{1/2}\sin\tfrac{1}{2}\theta \quad (R \to \infty), \\ &I = -\frac{2^{1/2}}{\pi}\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2} V(x)\,dx. \end{aligned} \tag{4.9}$$

Using the velocity V given in (4.1), we obtain

$$I = -2^{3/2}/\pi, \tag{4.10}$$

and if we expand for large R as in [10], we find that

$$\Phi_0 \sim -\frac{2^{3/2}}{\pi} R^{1/2} \sin \tfrac{1}{2}\theta + \frac{2^{1/2}}{\pi^2}\{(\pi - \theta) \cos \tfrac{1}{2}\theta + \sin \tfrac{1}{2}\theta \log R\} R^{-1/2}$$

$$+ \frac{2^{1/2}}{\pi^2}\{\log 4 + \gamma + 1 - i\pi\} R^{-1/2} \sin \tfrac{1}{2}\theta \quad (R \to \infty), \tag{4.11}$$

where γ is Euler's constant, together with a wave train. Now the outer solution ϕ must match with $\epsilon^{1/2}\Phi_0(R, \theta)$. Expanding (4.11) in terms of $\delta = \epsilon R$ we find terms of the order of $\epsilon \log \epsilon$ and ϵ.

To satisfy the matching condition (4.8) we therefore require

$$\phi \sim \phi_0 + \epsilon \log \epsilon \phi_1 + \epsilon \phi_2 + \cdots, \tag{4.12}$$

When this expansion is substituted into the full problem and like powers of ϵ are formally collected together, we get the following sequence of boundary-value problems for the outer potential:

$$\frac{\partial \phi_0}{\partial y} = V(x) \quad (|x| < 1), \qquad \phi_0 = 0 \quad (|x| > 1,\ y = 0),$$

$$\frac{\partial \phi_1}{\partial y} = 0 \quad (|x| < 1), \qquad \phi_1 = 0 \quad (|x| > 1,\ y = 0), \tag{4.13}$$

$$\frac{\partial \phi_2}{\partial y} = 0 \quad (|x| < 1), \qquad \phi_2 = -\frac{\partial \phi_0}{\partial y} \quad (|x| > 1,\ y = 0).$$

Each of these must also vanish at infinity. The solutions to these problems are easily found in terms of their complex potentials,

$$w_1(z) = -\tfrac{1}{2}iz^2 - \frac{z^2}{\pi}\log\frac{1 - i(z^2-1)^{1/2}}{z} + i\frac{1}{\pi}(z^2-1)^{1/2},$$

$$w_2(z) = Ai(z^2-1)^{-1/2}, \tag{4.14}$$

$$w_3(z) = -z + \frac{2i}{\pi^2}\frac{z}{(z^2-1)^{1/2}}\left(\log\frac{z-1}{z+1} - i\pi\right)$$

$$+ \frac{4i}{\pi^2}(z^2-1)^{1/2}\int_0^\infty \frac{\xi^2+1}{\xi^2-1}\tan^{-1}\xi \frac{d\xi}{(z-1)\xi^2 - (z+1)} + Bi(z^2-1)^{-1/2}.$$

The branch cuts between $z = \pm 1$ are chosen above the free surface, and the square roots and logarithms are chosen to be real and positive when z is large and real.

Note that w_2 is an eigensolution of the problem, and is therefore determined only up to two arbitrary constants. By the symmetry of the problem, we choose the constants to give an even solution, so that there is only one arbitrary constant A. Eigenfunctions $\mathrm{i}(z+1)^{n+1/2}/(z-1)^{n+1/2}$ for any integer n can be added to any of the above solutions, which is why w_3 is determined only up to the choice of B. It will become clear later in the matching procedure that only eigenfunctions with at worst $(z-1)^{-1/2}$ singularities can be matched at all to the inner solutions, so no further eigenfunctions are added to w_2 or w_3.

The matching condition (4.8) requires us to write the expression (4.12) in terms of R by the transformation $z=1+\epsilon R\,\mathrm{e}^{\mathrm{i}\theta}$ and expand formally up to some convenient order ϵ^m. Choosing $m=\frac{3}{2}$, and using the expansion for the integral in w_3 given in the Appendix,

$$\begin{aligned}\phi^{(1,3/2)} &= \epsilon^{1/2}\log\epsilon\left(A/2^{1/2}+2^{1/2}/\pi^2\right)R^{-1/2}\sin\tfrac{1}{2}\theta\\ &+\epsilon^{1/2}\left\{-\frac{2^{3/2}}{\pi}R^{1/2}\sin\tfrac{1}{2}\theta+\frac{1}{2^{1/2}}\left(B-\frac{2}{\pi^2}\log 2\right)R^{-1/2}\sin\tfrac{1}{2}\theta\right.\\ &\left.+\frac{2^{1/2}}{\pi^2}\left(\log R\sin\tfrac{1}{2}\theta+(\pi-\theta)\cos\tfrac{1}{2}\theta\right)R^{-1/2}\right\}\\ &+\epsilon\{R\sin\theta-1\}+\epsilon^{3/2}\log\epsilon\left(\frac{1}{4\cdot 2^{1/2}}A-\frac{11}{4}\frac{2^{1/2}}{\pi^2}\right)R^{1/2}\sin\tfrac{1}{2}\theta\\ &+\epsilon^{3/2}\left\{\frac{11}{4}\frac{2^{1/2}}{\pi^2}\left(\log R\sin\tfrac{1}{2}\theta+(\pi-\theta)\cos\tfrac{1}{2}\theta\right)R^{1/2}-2^{3/2}\frac{11}{12}\frac{1}{\pi}R^{3/2}\sin\tfrac{3}{2}\theta\right.\\ &\left.+\frac{1}{2^{1/2}}\left[\frac{1}{\pi^2}\left(1+\tfrac{3}{2}\log 2\right)+\tfrac{1}{4}B-\frac{2}{\pi^2}(2-q+\log 2)\right]R^{1/2}\sin\tfrac{1}{2}\theta\right\},\end{aligned}\tag{4.15}$$

where q is the constant

$$q=\frac{4}{\pi^2}\int_0^\infty\frac{\tan^{-1}s}{1-s^2}\,\mathrm{d}s.\tag{4.16}$$

Here the notation $\phi^{(1,3/2)}$ means that the expansion (4.13) was carried out to order ϵ^1, while expansion (4.16) is carried out to order $\epsilon^{3/2}$. A more detailed description of the matching procedure is given in [10] or [16].

Since we know that the function Φ that must be matched with (4.16) is of the order $\epsilon^{1/2}$, the coefficient of $\epsilon^{1/2}\log\epsilon$ must be zero. Therefore,

$$A=-2/\pi^2.\tag{4.17}$$

Also the form of (4.16) suggests an expansion for Φ in the form

$$\Phi\sim\epsilon^{1/2}\Phi_0+\epsilon\Phi_1+\epsilon^{3/2}\log\Phi_2+\epsilon^{3/2}\Phi_3+\cdots.\tag{4.18}$$

Substituting this expression into the full problem and formally equating powers of ϵ, we obtain the following sequence of problems:

$$\nabla^2\Phi_i = 0, \qquad \Phi_i = \frac{\partial\Phi_i}{\partial y} = 0 \quad (X>0,\ Y=0)\ (i=1,2,3),$$
$$\frac{\partial\Phi_1}{\partial y} = 1, \qquad \frac{\partial\Phi_j}{\partial y} = 0 \quad (j=2,3)\ (X<0,\ Y=0). \tag{4.19}$$

Also, the edge condition must be satisfied and conditions at infinity are provided by the rule $\Phi^{(3/2,1)} = \phi^{(1,3/2)}$, so that the limits of the potentials Φ_i $(i=0,\ldots,3)$ are given by each term in the expansion (4.15). For example, at order $\epsilon^{1/2}$, we see that

$$\phi_0 \sim -\frac{2^{3/2}}{\pi}R^{1/2}\sin\tfrac{1}{2}\theta + \frac{2^{1/2}}{\pi^2}(\log R\sin\tfrac{1}{2}\theta + (\pi-\theta)\cos\tfrac{1}{2}\theta)R^{-1/2}$$
$$+\frac{1}{2^{1/2}}\left(B - \frac{2}{\pi^2}\log 2\right)R^{-1/2}\sin\tfrac{1}{2}\theta. \tag{4.20}$$

Comparing with the known value in (4.11), we find

$$B = \frac{2}{\pi^2}(3\log 2 + \gamma + 1 - \mathrm{i}\pi). \tag{4.21}$$

The potential Φ_1 has a simple solution

$$\Phi_1 = R\sin\theta - 1, \tag{4.22}$$

and since

$$\Phi_2 \sim \frac{3}{2^{1/2}\pi^2}R^{1/2}\sin\tfrac{1}{2}\theta \quad \text{as } R\to\infty,$$

it is just a multiple of Φ_0,

$$\Phi_2 = \frac{3}{2\pi}\Phi_0. \tag{4.23}$$

The far-field behaviour of Φ_3 is the coefficient of $\epsilon^{3/2}$ in the expression (4.15), which we denote by $\hat{\Phi}_3$. If $\Phi_3 = \hat{\Phi}_3 + G$, using Holford's [6] method, G behaves at infinity like

$$G \sim \frac{-\mathrm{i}\exp(-\tfrac{1}{8}\mathrm{i}\pi - \mathrm{i}/\epsilon)}{4\pi^{3/2}}(-4\log 2 - 2\gamma + q - \mathrm{i}\pi - 36)\exp\{(\mathrm{i}x - y)/\epsilon\}. \tag{4.24}$$

Similarly, the wave amplitude of Φ_0 at infinity is

$$A_0 = \mathrm{i}I\epsilon^{1/2}(\pi/2)^{1/2}\exp(-\tfrac{1}{8}\mathrm{i}\pi - \mathrm{i}/\epsilon). \tag{4.25}$$

To summarize, the waves at infinity are given by

$$\Phi \sim A_+\exp\{(\mathrm{i}x - y)/\epsilon\} \quad (X\to\infty),$$

where

$$A_+ \sim -\mathrm{i}\pi^{-1/2}\exp(-\tfrac{1}{8}\mathrm{i}\pi - \mathrm{i}/\epsilon)\Big\{2\epsilon^{1/2} + \frac{3}{\pi}\epsilon^{3/2}\log\epsilon + \frac{1}{4\pi}(-4\log 2 - 2\gamma + q - 36 - \mathrm{i}\pi)\epsilon^{3/2}\Big\}. \tag{4.26}$$

In this problem $A_- = A_+$ by symmetry. Finally, the couple T exerted on each section is given by

$$\frac{1}{\rho l^3}T = \int_0^1 x\phi(x,0)\mathrm{d}x. \tag{4.27}$$

The range of integration includes both the inner and outer regions, but to order $\epsilon^{3/2}$ the inner region's contribution is negligible. At each order, we have the results

$$\int_0^1 x\phi_0(x,0)\mathrm{d}x = \frac{1}{24\pi}, \qquad \int_0^1 x\phi_1(x,0)\mathrm{d}x = -4/\pi^2,$$

$$\begin{aligned}\int_0^1 x\phi_2(x,0)\mathrm{d}x &= -\tfrac{1}{3} + \frac{2}{\pi^2}\int_0^1 \frac{x^2}{(1-x^2)^{1/2}}\log\frac{1-x}{1+x}\mathrm{d}x \\ &\quad + \frac{4}{\pi^2}\int_0^1 x(1-x^2)^{1/2}\int_0^\infty \frac{s^2+1}{s^2-1}\tan^{-1}s\frac{\mathrm{d}s}{(1-x)s^2-(x+1)}\mathrm{d}x + 2B \\ &= P_0 + \mathrm{i}P_1 \quad (\text{say}).\end{aligned}$$

Therefore,

$$\frac{1}{\rho l^3}T = \frac{1}{24\pi} - \frac{4}{\pi^2}\epsilon\log\epsilon + (P_0 + \mathrm{i}P_1)\epsilon + \cdots. \tag{4.28}$$

For a symmetric plate we thus find that the tuning conditions are

$$K = \Big\{\tfrac{1}{3}ml + \rho l^3\Big(\frac{1}{24\pi} - \frac{4}{\pi^2}\epsilon\log\epsilon + P_0\epsilon\Big)\Big\}(gl/\epsilon)^{1/2}, \qquad D = P_1\epsilon\rho l^3, \tag{4.29}$$

and the maximum efficiency is 50%.

We turn now to the full problem of the asymmetric raft, which moves so that the velocity is distributed over its surface as

$$\begin{aligned}V(x) &= \frac{x+1}{a+1} + \frac{2-(a+1)h}{a^2-1}(x-a)H(x-a) \\ &= \begin{cases}\dfrac{x+1}{a+1} & (-1<x<a), \\ \dfrac{(1-h)x+ah-1}{a-1} & (a<x<1).\end{cases}\end{aligned} \tag{4.30}$$

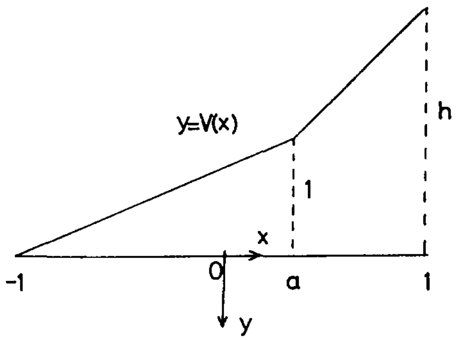


Figure 3. The asymmetric raft used in (4.37).

A sketch of this shape is given in Fig. 3, and note that a and h are the ratios of the lengths and amplitudes of oscillation of the left and right sections respectively. It is convenient also to let

$$a = \cos \psi . \tag{4.31}$$

The details of this are more complicated, but the principles are just the same as those above, so we give only an outline of the calculation and the difficulties. First the outer potentials which correspond to (4.14) are given by

$$\begin{aligned} w_1(z) = &-\frac{1}{2(a+1)} \mathrm{i}\left[(z+1)^2\left\{1-\left(\frac{z-1}{z+1}\right)^{1/2}\right\}-(z^2-1)^{1/2}\right] \\ &-\frac{1}{2\pi}\frac{2-(a+1)h}{a^2-1}\left[(z-a)^2 \log\frac{az-1+\mathrm{i}\sin\psi(z^2-1)^{1/2}}{a-z}\right. \\ &\left.-\mathrm{i}\psi\left(\frac{z-1}{z+1}\right)^{1/2}-\mathrm{i}(\sin\psi+\psi)(z^2-1)^{1/2}+\mathrm{i}\psi(1+a^2)\left(\frac{z-1}{z+1}\right)^{1/2}\right], \end{aligned} \tag{4.32a}$$

$$w_2(z) = \mathrm{i}\left\{B_1\left(\frac{z-1}{z+1}\right)^{1/2} + B_2\left(\frac{z+1}{z-1}\right)^{1/2}\right\}, \tag{4.32b}$$

and a complicated expression for w_3.

Notice that the eigenfunctions in w_2 and w_3 are not equally weighted. The constant B_1 must be determined by matching near $z = -1$ and B_2 by matching near $z = +1$.

In order to carry out the matching near the right inner region, we set $z = 1 + \delta$ and expand for small δ. This gives, to leading order only,

$$\begin{aligned} w_3 \sim \frac{\mathrm{i}}{\pi}&\left[\frac{1}{2(a+1)}\left(2^{1/2}+\frac{1}{2^{1/2}}\right)-\frac{1}{2\pi}-\frac{2-(a+1)h}{a^2-1}\left\{-\frac{(1-a)\sin\psi}{2^{1/2}}+\frac{a\psi}{a^{1/2}}\right.\right. \\ &\left.\left.-\frac{1}{2^{1/2}}(\sin\psi-\psi)\right\}\right]\delta^{-1/2}\log\delta = \frac{\mathrm{i}}{\pi}\tilde{B}_2\delta^{-1/2}\log\delta \quad \text{(say)}. \end{aligned} \tag{4.33}$$

Similarly, near the left inner region, we set $z = -1 + \delta$ and expand for small δ to obtain

$$w_3 \sim \frac{\mathrm{i}}{\pi}\left[\frac{1}{2(a+1)}\frac{\mathrm{i}}{2^{1/2}} + \frac{\mathrm{i}}{2\pi}\frac{2-(a+1)h}{a^2-1}\left\{\frac{(1+a)\sin\psi}{2^{1/2}}\right.\right.$$

$$\left.\left. + \frac{1}{2^{1/2}}(\sin\psi + \psi)\right\}\right]\delta^{-1/2}\log\delta = -\frac{1}{\pi}\tilde{B}_1\delta^{-1/2}\log\delta \quad \text{(say)}. \tag{4.34}$$

The same argument that was used to derive the constant A for the symmetric raft now gives

$$B_1 = \frac{1}{2^{1/2}\pi}\tilde{B}_1, \qquad B_2 = \frac{1}{2^{1/2}\pi}\tilde{B}_2. \tag{4.35}$$

the evaluation of C_1 and C_2 is, however, prohibitively complicated and so we keep expansions only up to order $\epsilon \log \epsilon$. The constants I_1 and I_2 corresponding to I for the symmetric dock can either be evaluated from Holford's [6] integral or by examining w_1 near $z = \pm 1$. Either way,

$$-\frac{\pi}{2^{1/2}}I_1 = \int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{1/2}V(x)\mathrm{d}x = \frac{1}{a+1}\left(\frac{\pi}{2} - \tfrac{1}{2}\psi + \tfrac{1}{4}\sin 2\psi\right)$$

$$+ \frac{ah-1}{a-1}(\psi - \sin\psi) + \frac{1-h}{a-1}\left(-\tfrac{1}{4}\sin\psi + \sin\psi - \tfrac{1}{2}\psi\right), \tag{4.36}$$

$$-\frac{\pi}{2^{1/2}}I_2 = \int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2}V(x)\mathrm{d}x = \frac{1}{a+1}\left\{\tfrac{3}{2}(\pi-\psi) - 2\sin\psi - \tfrac{1}{4}\sin 2\psi\right\}$$

$$+ \frac{1-h}{a-1}\left(\tfrac{1}{2}\psi + \sin\psi + \tfrac{1}{4}\sin 2\psi\right) + \frac{ah-1}{a-1}(\psi + \sin\psi). \tag{4.37}$$

Using the results (4.36)–(4.38), we see that

$$A_+ = -\mathrm{i}\pi^{-1/2}\exp\left(-\tfrac{1}{8}\mathrm{i}\pi - \mathrm{i}/\epsilon\right)\left\{-\pi/2^{1/2}I_2\epsilon\right.$$

$$\left. + \frac{1}{2\pi}\left(-\frac{\pi}{2^{1/2}}I_1 - 2J_2\right)\epsilon^{3/2}\log\epsilon\right\}, \tag{4.38a}$$

$$A_- = -\mathrm{i}\pi^{-1/2}\exp\left(-\tfrac{1}{8}\mathrm{i}\pi - \mathrm{i}/\epsilon\right)\left\{-\pi/2^{1/2}I_1\epsilon\right.$$

$$\left. + \frac{1}{2\pi}\left(-\frac{\pi}{2^{1/2}}I_2 - 2J_1\right)\epsilon^{3/2}\log\epsilon\right\},$$

where

$$J_1 = \frac{1}{a+1}(\pi - \psi) + \frac{ah-1}{a-1}\tan\psi + \frac{1-h}{a-1}(\psi - \tan\tfrac{1}{2}\psi),$$
$$J_2 = \frac{1}{a+1}\left\{\frac{2-h}{\tan\frac{1}{2}\psi} - (\pi - \psi)\right\} + \frac{1-h}{a-1}(\tfrac{1}{2}\psi - \tfrac{1}{4}\sin 2\psi). \tag{4.38b}$$

A useful check on our derivation of J_1 and J_2 is given by Leppington [10] where expressions are given as integrals of V rather than as derivations from B_1 and B_2. In principle, the added-mass and damping coefficients are given by integrals of the potentials in (4.32), but these are very hard to calculate explicitly.

The maximum efficiency is also quite a complicated expression, but in the special case $h = 1$, where one section of the raft is held horizontal, it simplifies. Then to leading order, using (4.38),

$$E_M = \tfrac{1}{2} + \frac{\psi \sin\psi}{\psi^2 + \sin^2\psi}. \tag{4.39}$$

A slightly surprising implication is that when $\psi = 0$, we have 100% efficiency in this limit. This corresponds to a single plate clamped at one end. Furthermore, increasing ψ decreases the efficiency. Of course, this holds only in the limit $\epsilon \to 0$ and more complicated effects occur if, say, both terms in (4.32) are kept.

(b) The long-wave limit

In the long-wave limit where $kl \to 0$, it is usually best to use a Green function method and to solve by iteration. The integral equation is set up in much the same way as for the vertical barrier, and the method is essentially due to McCamy [12]. We define the reduced potential right at the start as

$$\frac{\partial\theta}{\partial y} = \frac{\partial\phi}{\partial y} + k\phi. \tag{4.40}$$

Again the problem is non-dimensionalised by dividing by l, so that $(kl, l, x, y) \to (k, 1, x/l, y/l)$.

Following McCamy [12], this gives the integral equation

$$f(\xi) - \frac{k}{\pi}\int_{-1}^{1} f(x)\log|x-\xi|\,dx = U(\xi) \quad (|\xi| < 1), \tag{4.41a}$$

where

$$U(x) = V(x) + k^2\int_0^x\int_0^{x_1} V(x_2)\,dx_2\,dx_1 + k(A + Bx) \tag{4.41b}$$

and

$$f(x) = \left.\frac{\partial\theta}{\partial y}\right|_{y=0} = k\phi(x,0) + \phi_y(x,0). \tag{4.41c}$$

This is a version of the aerofoil equation of the second kind. Although the aerofoil equation has an explicit solution, there seems to be no simple solution to (4.41). However, it is well suited to iteration for small k because the integral is small in that limit.

It remains to evaluate the constants A and B, which arise because we are working with the reduced rather than the full potential. Using (4.41a) and (4.41b), it can be shown that

$$A=\int_0^1\{f(-x)+f(x)\}g_1(x)\mathrm{d}x \tag{4.42}$$

and

$$B=\int_0^1\{f(-x)-f(x)\}g_1'(x)\mathrm{d}x, \tag{4.43}$$

where

$$g_1(|x-\xi|)=g(x,\xi)-\frac{1}{\pi}\log|x-\xi|$$
$$=\mathrm{i}\mathrm{e}^{\mathrm{i}k|x-\xi|}-\frac{k}{2\pi}\int_0^\infty\log\left[(x-\xi)^2+t^2\right]\mathrm{e}^{-kt}\mathrm{d}t$$

and

$$g(x,\xi)=g(x,0;\xi,0)=-\mathrm{i}\mathrm{e}^{\mathrm{i}k|x-\xi|}-\frac{1}{\pi}\int_0^\infty\frac{t}{1+t^2}\mathrm{e}^{k|x-\xi|t}\mathrm{d}t.$$

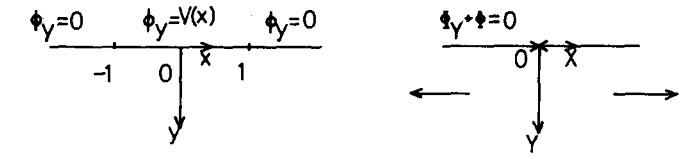
McCamy [12] shows that (4.41)–(4.43) give an implicit scheme which guarantees a solution.

Now we seek a series solution, in k, to this implicit scheme. Writing out (4.41) in full by using (4.43) and (4.42),

$$f(\xi)-\frac{k}{\pi}\int_{-1}^1 f(x)\log|x-\xi|\mathrm{d}x$$
$$=V(\xi)+k^2V_2(\xi)$$
$$+\left\{-\mathrm{i}k\int_{-1}^1 f(x)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x+\frac{k\log k}{\pi}\int_{-1}^1 f(x)\mathrm{d}x\right.$$
$$\left.-\frac{k}{\pi}\int_{-1}^1 f(x)\int_0^\infty\log\left[(x^2+t^2)^{1/2}\right]\mathrm{e}^{-kt}\mathrm{d}t\mathrm{d}x\right\}$$
$$+k^2\left\{\int_{-1}^1 f(x)\,\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x-\frac{k}{\pi}\int_{-1}^1\int_0^\infty\frac{x}{x^2+t^2}\mathrm{e}^{-kt}\mathrm{d}t\mathrm{d}x\right\}. \tag{4.44}$$

The form of (4.44) suggests an expansion for f in the form

$$f=f_0+k\log k f_1+kf_2+(k\log k)^2f_3+k^2\log k f_4+k^2f_5+\cdots, \tag{4.45}$$



(a) (b)

Figure 4. Inner, (a), and outer, (b), problems for the dock in the long-wave limit.

and substituting this into (4.44) yields

$$f_0(\xi)=V(\xi), \tag{4.46a}$$

$$f_1=\frac{1}{\pi}\int_{-1}^{1}V(x)\mathrm{d}x=\frac{1}{\pi}P \quad \text{(say)}, \tag{4.46b}$$

$$f_2(\xi)=\frac{1}{\pi}\int_{-1}^{1}V(x)\log|x-\xi|\mathrm{d}x-\left(\frac{\gamma}{\pi}+\mathrm{i}\right)P, \tag{4.46c}$$

$$f_3=\frac{1}{\pi}\int_{-1}^{1}f_1(x)\mathrm{d}x=\frac{2}{\pi^2}P, \tag{4.46d}$$

where γ is Euler's constant. In principle, this expansion procedure could be continued indefinitely, but for the moment consider just the first four terms.

A simple check on the results (4.46) is possible using a matching scheme. Suppose $V=1$, and we scale the length l out of the problem as in Fig. 4. The full problem is then

$$\begin{aligned}&\nabla^2\phi=0 \quad (y>0),\\ &\phi_y+k\phi=0 \quad (|x|>1), \qquad \phi_y=1 \quad (|x|<1,\ y=0).\end{aligned} \tag{4.47}$$

At $O(1)$ distances from the dock, the waves are so long they are invisible, and we simply set $k=0$ in (4.47). This gives an inner solution

$$w_1(z)=A-\frac{1}{\pi}\left\{z\log\frac{z+1}{z-1}+\log(z^2-1)\right\}, \tag{4.48}$$

where A is a constant to be determined. This solution does not take into account the waves at infinity which are propagated outwards. However, far from the dock we set $X=kx$, $Y=ky$, $Z=X+\mathrm{i}Y$, and the length of the dock appears small. We can therefore model it as a floating point source of water-waves of unknown strength B, causing an outer potential

$$\begin{aligned}\Omega_1(Z)&=B\int_\Gamma\cos Xt\,\mathrm{e}^{Yt}\frac{\mathrm{d}t}{t-1}\\ &=B\left[\tfrac{1}{2}(E_1(-\mathrm{i}Z)+E_1(\mathrm{i}\bar{Z})+\begin{cases}\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\bar{Z}} & (X>0)\\ \pi\mathrm{i}\mathrm{e}^{-\mathrm{i}Z} & (X<0)\end{cases}\right].\end{aligned} \tag{4.49}$$

To perform the matching, let $z \to \infty$ in (4.48) and $Z \to 0$ in (4.49), giving

$$w_1 \sim A + \frac{2}{\pi} - \frac{2}{\pi}\log z \quad (z \to \infty),$$

$$\Omega_1 \sim -B\{\log Z + \gamma + \pi\} \quad (Z \to 0).$$

Replacing z by Z/k and matching terms, we obtain

$$B = \frac{2}{\pi}, \qquad A = -\frac{2}{\pi}(\log k + \gamma + 1 + \pi\mathrm{i}). \tag{4.50}$$

In particular, on the dock,

$$w_1 = -\frac{1}{\pi}\left\{z\log\frac{z+1}{z-1} + \log(z^2 - 1) + 2(\log k + \gamma + 1 + \pi\mathrm{i})\right\}, \tag{4.51}$$

and $f(x) = \theta_y(x, 0) = k\phi(x, 0) + V(x)$. If this is compared with the results in (4.46) for $f_0 - f_2$, they are found to agree.

Applying Green's theorem to ϕ and $\mathrm{e}^{-ikx-ky}$ along the contour S_H in Fig. 2 in the usual way, and letting $H \to \infty$,

$$A_\pm = \mathrm{i}\int_{-1}^{1} V(x)\,\mathrm{e}^{\pm ikx}\mathrm{d}x + \mathrm{i}k\int_{-1}^{1}\phi(x,0)\,\mathrm{e}^{\pm ikx}\mathrm{d}x = \mathrm{i}\int_{-1}^{1} f(x)\,\mathrm{e}^{\pm ikx}\mathrm{d}x. \tag{4.52}$$

By using the expansion (4.45), and the asymmetric raft shape $V(x)$ in (4.30), this gives the first two terms of the expansion for $A_\pm$ as

$$A_\pm = \mathrm{i}\{1 + \tfrac{1}{2}h(1-a)\}\left\{1 + \frac{2}{\pi}k\log k + O(k)\right\}. \tag{4.53}$$

The successive terms are complicated because expressions like (4.32) have to be integrated. Notice that to $O(k)$ the waves at infinity are symmetric, so the maximum efficiency in this limit is 50% + $O(k)$. The added mass M and the damping N are similarly found to be

$$M + \mathrm{i}N = \rho l\int_{-1}^{1}\{f(x) - V(x)\}\mathrm{d}x. \tag{4.54}$$

Again using (4.46), this gives, to leading order,

$$M = \frac{2\rho l}{\pi}\{1 + \tfrac{1}{2}h(1-a)\}\log k, \qquad N = 2\rho l\{1 + \tfrac{1}{2}h(i-a)\}. \tag{4.55}$$

An interesting feature of this result is that the added mass is logarithmically large in this limit, although the radiation damping remains finite. This is typical of many two-dimensional radiation problems.

5. Moving-barrier devices

Elsewhere [14], a theory was developed for diffraction by plane obstacles which are nearly vertical. If a wave is incident on a flexible plate of length a hinged at the surface and hanging vertically, then it will oscillate with the frequency of the incoming wave. Its shape will then be

$$x = c(y, t) = \xi c(y)\, \mathrm{e}^{\mathrm{i}\omega t}. \tag{5.1}$$

Here ξ is a complex number since the oscillation may be out of phase with the incident field. The boundary condition for ϕ on the body, linearised for small ξ, is

$$\frac{\partial \phi}{\partial x} = \mathrm{i}\xi\omega c(y) \quad (x = 0,\ -a < y < 0). \tag{5.2}$$

Very briefly, we now summarise the method used. At first order in ξ an integral equation is obtained for ϕ_1, where $\phi = \phi_1 + \xi\phi_2$,

$$\phi_1 = \phi_0 + \frac{1}{2\pi}\int_{-a}^{0} f_1(y)(G_x)_{x=0}\mathrm{d}y.$$

The function $f(y) = f_1(y) + \xi f_2(y)$ is the discontinuity of ϕ across the barrier, and G is the usual Green function for water-waves. This can be solved for f_1 by applying the boundary conditions and solving the resulting equation,

$$2\pi\, \mathrm{e}^{k y_1} = \frac{\partial^2}{\partial y_1^2}\int_{-a}^{0} f_1(y)(G)_{x=x_1=0}\mathrm{d}y.$$

This gives the result

$$f_1(y) = C_1\, \mathrm{e}^{k y}\int_{-a}^{y} \mathrm{e}^{-ku} p_1'(u)\mathrm{d}u, \tag{5.3}$$

where

$$p_1(u) = (a^2 - u^2)^{1/2}, \qquad C_1 = 2\mathrm{i}[K_1(ka) - \mathrm{i}\pi I_1(ka)]^{-1}.$$

At second order we get the equation

$$\mathrm{i}\omega c(y_1) = -\frac{\partial}{\partial y_1}\left(c(y_1)\frac{\partial}{\partial y_1}\mathrm{e}^{k y_1}\right) + \frac{1}{2\pi}\frac{\partial^2}{\partial y_1^2}\int_{-a}^{0} f_2(y)(G)_{x=x_1=0}\mathrm{d}y.$$

If we let $g(y) = \int_0^y c(p)\mathrm{d}p$ and

$$\begin{aligned} M(x) &= \int_0^{-x} c'(ap)\, \mathrm{e}^{kap}\mathrm{d}p + \mathrm{i}\omega\int_0^{-x}(ka(g(ap)) - g'(xp))\mathrm{d}p \\ &= M_1(x) + \mathrm{i}\omega M_2(x) \quad \text{(say)}, \end{aligned} \tag{5.4}$$

then the solution for f_2 is

$$f_2(y) = e^{ky}\int_{-a}^{y} e^{-ku}p'(u)\,du,$$

where

$$p(s) = 2C_2 ka(1-s^2)^{1/2} + \frac{4ka}{\pi}(1-s^2)^{1/2}\int_0^1 \frac{xM(x)}{(1-x^2)^{1/2}(x^2-s^2)}\,dx$$

$$= 2C_2 kap_1(s) + \frac{4ka}{\pi}p_2(s) \quad \text{(say)}. \tag{5.5}$$

Here,

$$C_2 = -\frac{2}{\pi^2}\frac{\mu - i\pi\lambda}{K_1 - iI_1},$$

where the argument of all the functions is ka and

$$\mu(z) = \int_{-1}^{1} p_2'(s)\int_0^\infty \frac{e^{-zu}}{u+s}\,du\,ds, \qquad \lambda(z) = \int_{-1}^{1} e^{-zs}p_2'(s)\,ds. \tag{5.6}$$

We also write μ_1 and λ_1 for the expressions obtained when M is replaced by M_1 in (5.5), and similarly for μ_2 and λ_2. Notice that the solution for ϕ_2 has divided into two parts. The first, corresponding to $M = M_1$, is the effect of a stationary but curved barrier and is in phase with the motion of the barrier itself. The second, corresponding to $M = M_2$, is the effect of the motion and is 90° out of phase with the barrier's movement.

The reflection and transmission coefficients are given by $R = R_1 + \xi R_2$ and $T = T_1 + \xi T_2$ respectively, where

$$\begin{aligned} R_1 &= \frac{1}{2i}C_1K_1(ka), & T_1 &= -\frac{1}{2i}C_1I_1(ka), \\ R_2 &= P + Q, & T_2 &= P - Q, \end{aligned} \tag{5.7}$$

and

$$P = \frac{ka}{K_1(ka) - i\pi I_1(ka)}\left\{ka\int_{-1}^{0} c(y)\,e^{-2kay}\left(\int_{-1}^{y} e^{-kau}\frac{u}{(1-u^2)^{1/2}}\,du\right)dy \right.$$

$$\left. + \int_{-1}^{0} c(y)\,e^{kay}\frac{y}{(1-y^2)^{1/2}}\,dy\right\}, \tag{5.8}$$

$$Q = \frac{2}{\pi}\frac{\lambda(ka)K_1(ka) - \mu(ka)I_1(ka)}{K_1(ka) - i\pi I_1(ka)}.$$

Note that, in this case, P and Q are not necessarily equal, as for a stationary barrier.

Now the restoring force on the plate due to the fluid is

$$F=\mathrm{i}\omega\rho\int_c \phi\cos(\boldsymbol{n},\boldsymbol{i})\mathrm{d}s=\mathrm{i}\omega\rho\int_{-1}^{0} f(y)\mathrm{d}y, \tag{5.9}$$

by linearising $\cos(\boldsymbol{n}, \boldsymbol{i})$ for small movements. At first order, we simply obtain

$$\int_{-1}^{0} f_1(y)\mathrm{d}y=C_1\int_{-0}^{1}\mathrm{e}^{kau}(1-u^2)^{1/2}\mathrm{d}u$$

$$=\psi(ka)=\frac{2\pi\mathrm{i}}{ka}\frac{I_1(ka)+L_1(ka)}{K_1(ka)-\mathrm{i}\pi I_1(ka)}, \tag{5.10}$$

and this is the well-known result for the force on a vertical barrier [7]. Similarly, define

$$\int_{-a}^{0} f_2(y)\mathrm{d}y=X(ka)=X_1(ka)+\mathrm{i}\omega X_2(ka), \tag{5.11}$$

where X_1 is obtained by setting $M=M_1$ and X_2 by setting $M=M_2$. If the barrier is mounted on a spring-and-damper-type of system (2.5), then the equation of motion of the plate gives

$$\mathrm{i}\omega\rho\{\psi+\xi(X_1+\mathrm{i}\omega X_2)\}=(1-m\omega^2+D\mathrm{i}\omega+K)\xi. \tag{5.12}$$

Finally, using Evans' [2] method, we obtain an efficiency of

$$E=\frac{2\omega^2 D}{\rho g^2}|\xi|^2=\frac{(2\omega^5/g)D|\psi|^2}{|K+\mathrm{i}\omega(D-\rho X_1)-\omega^2(m-\rho X_2)|^2}. \tag{5.13}$$

This shows that the maximum efficiency is

$$E_{\mathrm{M}}=(2\omega^5/g)|\psi|^2, \tag{5.14}$$

which is just the efficiency of a flat barrier, and to obtain this we require the tuning conditions

$$K=m\omega^2-\rho X_{2_r}\omega^2-\rho X_{1_i}\omega, \tag{5.15}$$

$$D=\frac{1}{\omega}\rho X_{2_r}-\rho X_{1_r}. \tag{5.16}$$

Consider next a shape which is symmetrical about $x=0$, and is slender but not flat. Then the two sides of the body are given by

$$x=\pm c(y,t)=\pm\xi c(y)\,\mathrm{e}^{\mathrm{i}\omega t}\quad(-a<y<0). \tag{5.17}$$

The boundary condition on the body, linearised for small amplitudes $|\xi|$, is

$$\frac{\partial\phi}{\partial x} = \pm i\xi\omega c(y), \tag{5.18}$$

where the time factor $e^{i\omega t}$ has been suppressed.

To first order, the problem is simply that of a stationary, vertical barrier and so the result (5.3) holds. At second order, applying (5.18) on $x = \pm 0$ gives

$$\pm i\omega c(y_1) = \pm\frac{1}{2\pi}\frac{\partial^2}{\partial y_1^2}\int_{-a}^{0} f_2(y)(G)_{x=x_1=0}\,dy \pm \tfrac{1}{2}D_1' f_1(y). \tag{5.19}$$

Here f_1 is given in (5.3) and the operator D_1' is given by

$$D_1' = \frac{\partial}{\partial y_1}\left(c(y_1)\frac{\partial}{\partial y_1}\right). \tag{5.20}$$

This is the same form of equation as before, except that this time the function M in (5.4) is

$$\begin{aligned} M(x) &= \tfrac{1}{2}C_1\int_0^{-x} kac'(y)\,e^{ky}\left\{\int_{-a}^{y} e^{-ku}p_1'(u)\,du - c(y)p_1'\right\}dy + i\omega M_2(x) \\ &= \tfrac{1}{2}C_1 M_3(x) + i\omega M_2(x) \quad \text{(say)}, \end{aligned} \tag{5.21}$$

where M_2 is just as in (5.4). The corresponding value for C_2, as in (5.6), is

$$C_2 = -\frac{2}{\pi^2}\frac{\mu - \pi\lambda}{K_1 - i\pi I_1} = -\frac{2}{\pi^2}(K_1 - i\pi I_1)^{-1}\{\tfrac{1}{2}C_1(\mu_3 - i\pi\lambda_3) + i\omega(\mu_2 - i\pi\lambda_2)\}. \tag{5.22}$$

All functions have argument ka, and λ_3 and μ_3 are obtained by setting $M = M_3$ in the definitions (5.5)–(5.7). The corrections to the reflection and transmission coefficients R_2 and T_2 corresponding to (5.8) are given by

$$R_2 = P + Q, \qquad T_2 = P - Q,$$

where

$$P = -2i(ka)^2\int_{-1}^{0} c(y)\,e^{2kay}\,dy, \tag{5.23}$$

and Q is as in (5.9), but with the M in (5.21). To complete the calculations, we also require the compression force on the shape. Using the theory given earlier,

$$\begin{aligned} \phi_1 + \xi\phi_2 &= \phi_0 + \frac{1}{2\pi}\int_{-a}^{0} f_1(y)(G_x)_{x=0}\,dy \\ &\quad + \xi\left\{\frac{1}{2\pi}\int_{-a}^{0} f_2(y)(G_x)_{x=0}\,dy - \frac{1}{\pi}\int_{-a}^{0}\phi_0(0, y)(D_1 G)_{x=0}\,dy\right\}. \end{aligned} \tag{5.24}$$

Now as $x \to \pm 0$, since $g(y) = \phi_0(0, y) = e^{ky}$,

$$\int_{-a}^{0} g(y)(D_1 G)_{x=0} \mathrm{d}y$$

$$\sim -2k\int_{-a}^{0} e^{ky}c(y)\left\{\frac{1}{y-y_1} + \frac{1}{y+y_1} - 2k\left(e^{-ky}E_1(ky) + \pi i\right)\right\}\mathrm{d}y$$

$$= f_3(y_1) \quad \text{(say)}. \tag{5.25}$$

Also,

$$\frac{1}{2\pi}\int_{-a}^{0} f(y)(G_x)_{x=0}\mathrm{d}y \to \pm\tfrac{1}{2}f(y) \quad (x_1 \to \pm 0). \tag{5.26}$$

Now define

$$F_j = \int_{-a}^{0} f_j(y)\mathrm{d}y \quad (j = 0, \ldots, 3),$$

so that the net compression force is

$$2(F_0 - \xi F_3) = 2\int_{-1}^{0}\{\phi_0(0, y) - \xi f_3(y)\}\mathrm{d}y. \tag{5.27}$$

To find an equation for ξ, we require an equation of motion for the walls of the bag. Suppose, for example, that the space between the walls is filled with gas at a pressure P satisfying

$$PV = K, \tag{5.28}$$

where V is the volume of the bag. Then

$$P = \frac{K}{V} = K \,\mathrm{Re}\frac{1}{\xi A\, e^{i\omega t}} = \frac{K}{A_0}\mathrm{Re}\frac{1}{\bar{\xi}}e^{i\omega t}, \quad A_0 = 2\int_{-a}^{0} c(y)\mathrm{d}y. \tag{5.29}$$

Using (5.27), and eliminating $e^{i\omega t}$,

$$a(F_0 - \xi F_3) = \frac{K}{A_0}\frac{1}{\bar{\xi}}$$

$$\Rightarrow |\xi|^4|F_3|^2 + \left(\frac{2K}{aA_0}\mathrm{Re}\,F_3 - |F_0|^2\right)|\xi|^2 + \left(\frac{K}{aA_0}\right)^2 = 0. \tag{5.30}$$

This quadratic can now, in principle, be solved for $|\xi|^2$, and once this is known the efficiency is equal to $2\omega^3 K/(\rho g^2 aA_0)|\xi|^2$.

Appendix

In Section 4, to derive (4.16) it is necessary to estimate the integral

$$I=\int_0^\infty \frac{1}{ax^2+b}\tan^{-1}x\,\mathrm{d}x \tag{A.1}$$

as $a \to 0$. The problem is that this is a singular limit. If a is set to zero, the integral does not converge. Letting $a^{1/2}x=u$,

$$I=a^{-1/2}\int_0^\infty \frac{1}{u^2+b}\tan^{-1}\left(\frac{u}{a^{1/2}}\right)\mathrm{d}u=\frac{\pi}{4(ab)^{1/2}}-\frac{1}{b^{1/2}}\int_0^\infty \tan^{-1}\left(\frac{u}{b^{1/2}}\right)\frac{\mathrm{d}u}{u^2+a} \tag{A.2}$$

by integrating by parts. We now choose an $\epsilon>0$ such that $a^{1/2} \ll \epsilon \ll 1$. The precise value of ϵ is not specified at this stage. Then divide the range of integration of the integral on the right-hand side of (A.2),

$$\int_0^\infty \tan^{-1}\left(\frac{u}{b^{1/2}}\right)\frac{\mathrm{d}u}{u^2+a}=\int_0^\epsilon \tan^{-1}\left(\frac{u}{b^{1/2}}\right)\frac{\mathrm{d}u}{u^2+a}+\int_\epsilon^\infty \tan^{-1}\left(\frac{u}{b^{1/2}}\right)\frac{\mathrm{d}u}{u^2+a}. \tag{A.3}$$

In the first integral, $u \ll 1$ so $\tan^{-1}(u/b^{1/2})$ can be expanded for small u. In the second, $(u^2/a)>(\epsilon^2/a)\gg 1$ and so $(u^2+a)^{-1}$ can be expanded. This gives that (A.3) is

$$\int_0^\epsilon \frac{u/b^{1/2}}{u^{1/2}+a}\mathrm{d}u+\int_\epsilon^\infty \frac{\tan^{-1}(u/b^{1/2})}{u^2}\mathrm{d}u+O(\epsilon^2/b)+O(a/\epsilon^2)$$

$$=\frac{1}{b^{1/2}}\left\{\tfrac{1}{2}\log(a+\epsilon^2)-\tfrac{1}{2}\log a+\left[-\frac{1}{q}\tan^{-1}q\right]_{\epsilon/b^{1/2}}^{\infty}+\int_\infty^{\epsilon/b^{1/2}}\frac{\mathrm{d}q}{q(1+q^2)}\right\}$$

$$=\frac{1}{b^{1/2}}\left\{\log\epsilon-\tfrac{1}{2}\log a+1-\log\epsilon+\tfrac{1}{2}\log b\right\}+O(\epsilon^2/b,\ a/\epsilon^2). \tag{A.4}$$

The $\log\epsilon$ terms cancel, as they must, since the choice of ϵ is left open. Using (A.4) in (A.2) now gives

$$I=\frac{\pi^2}{4b^{1/2}}a^{-1/2}+\frac{1}{2b}\log a-\frac{1}{b}\left(1+\tfrac{1}{2}\log b\right)+O(a/b,\ a). \tag{A.5}$$

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